

SIMPLIFICATION OF THE MAJORIZING MEASURES METHOD, with development.

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ABSTRACT.

We update, specify, review and develop in this article the classical majorizing measures method for investigation of the local structure of random fields, belonging to X.Fernique and M.Talagrand in order to simplify and improve the constant values.

Our considerations based on the generalization of the L.Arnold and P.Imkeller generalization of classical Garsia - Rodemich - Rumsey inequality.

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1 Notations. Statement of problem.

Let (X, d) , (Y, ρ) be separable metric spaces, m be arbitrary distribution, i.e. Radon probabilistic measure on the set X , $f : X \rightarrow Y$ be (measurable) function. Let also $\Phi(z)$, $z \geq 0$ be continuous Young-Orlicz function, i.e. strictly increasing function such that

$$\Phi(z) = 0 \Leftrightarrow z = 0; \lim_{z \rightarrow \infty} \Phi(z) = \infty.$$

We denote as usually

$$\Phi^{-1}(w) = \sup\{z, z \geq 0, \Phi(z) \leq w\}, w \geq 0$$

the inverse function to the function Φ ;

$$B(r, x) = \{x_1 : x_1 \in X, d(x_1, x) \leq r\}, x \in X, 0 \leq r \leq \text{diam}(X)$$

be the closed ball of radii r with center at the point x .

Let us introduce the Orlicz space $L(\Phi) = L(\Phi; m \times m, X \otimes X)$ on the set $X \otimes X$ equipped with the Young - Orlicz function Φ .

We assume henceforth that for all the values $x_1, x_2 \in X$, $x_1 \neq x_2$ (the case $x_1 = x_2$ is trivial) the value $\rho(f(x_1), f(x_2))$ belongs to the space $L(\Phi)$.

Note that for the existence of such a function $\Phi(\cdot)$ is necessary and sufficient only the integrability of the distance $\rho(f(x_1), f(x_2))$ over the product measure $m \times m$:

$$\int_X \int_X \rho(f(x_1), f(x_2)) m(dx_1) m(dx_2) < \infty,$$

see [23], chapter 2, section 8.

Under this assumption the distance $d = d(x_1, x_2)$ may be constructively defined by the formula:

$$d_\Phi(x_1, x_2) := \|\rho(f(x_1), f(x_2))\| L(\Phi). \quad (1.1)$$

Since the function $\Phi = \Phi(z)$ is presumed to be continuous and strictly increasing, it follows from the relation (1.1) that $V(d_\Phi) \leq 1$, where by definition

$$V(d) := \int_X \int_X \Phi \left[\frac{\rho(f(x_1), f(x_2))}{d(x_1, x_2)} \right] m(dx_1) m(dx_2). \quad (1.2)$$

Let us define also the following important distance function: $w(x_1, x_2) =$

$$w(x_1, x_2; V) = w(x_1, x_2; V, m) = w(x_1, x_2; V, m, \Phi) = w(x_1, x_2; V, m, \Phi, d) \stackrel{def}{=}$$

$$6 \int_0^{d(x_1, x_2)} \left\{ \Phi^{-1} \left[\frac{4V}{m^2(B(r, x_1))} \right] + \Phi^{-1} \left[\frac{4V}{m^2(B(r, x_2))} \right] \right\} dr, \quad (1.3)$$

where $m(\cdot)$ is probabilistic Borelian measure on the set X .

The triangle inequality and other properties of the distance function $w = w(x_1, x_2)$ are proved in [24].

Definition 1.1. (See [24]). The measure m is said to be *minorizing measure* relative the distance $d = d(x_1, x_2)$, if for each values $x_1, x_2 \in X$ $V(d) < \infty$ and moreover $w(x_1, x_2; V(d)) < \infty$.

We will denote the set of all minorizing measures on the metric set (X, d) by $\mathcal{M} = \mathcal{M}(\mathcal{X})$.

Evidently, if the function $w(x_1, x_2)$ is bounded, then the minorizing measure m is majorizing. Inverse proposition is not true, see [24], [1].

Remark 1.1. If the measure m is minorizing, then

$$w(x_n, x; V(d)) \rightarrow 0 \Leftrightarrow d(x_n, x) \rightarrow 0, \quad n \rightarrow \infty.$$

Therefore, the continuity of a function relative the distance d is equivalent to the continuity of this function relative the distance w .

Remark 1.2. If

$$\sup_{x_1, x_2 \in X} w(x_1, x_2; V(d)) < \infty,$$

then the measure m is called *majorizing measure*. This classical definition with theory explanation and applications basically in the investigation of local structure of random processes and fields belongs to X.Fernique [10], [11], [12] and M.Talagrand [40], [41], [42], [43], [44]. See also [3], [4], [5], [9], [25], [30], [31], [32].

The following important inequality belongs to L.Arnold and P.Imkeller [1], [19]; see also [21], [2].

Theorem of L.Arnold and P.Imkeller. *Let the measure m be minorizing. Then there exists a modification of the function f on the set of zero measure, which we denote also by f , for which*

$$\rho(f(x_1), f(x_2)) \leq w(x_1, x_2; V, m, \Phi, d). \quad (1.4)$$

As a consequence: this function f is d -continuous and moreover w -Lipshitz continuous with unit constant.

The inequality (1.4) of L.Arnold and P.Imkeller is significant generalization of celebrated Garsia - Rodemich - Rumsey inequality, see [15], with at the same applications as mentioned before [18], [30], [31], [32], [38].

Our purpose in this report is application of the L.Arnold and P.Imkeller inequality to the investigation of continuity of random fields; limit theorems, in particular, Central Limit Theorem, for random processes; exponential estimates for distribution of maximum of random fields etc.

Obtained results improve and generalize recent ones in [4], [9], [10], [18], [28], [33], [38], [41], [43] etc.

Remark 1.3. The inequality of L.Arnold and P.Imkeller (1.4) is closely related with the theory of fractional order Sobolev's - rearrangement invariant spaces, see [2], [15], [18], [21], [27], [31], [38], [39].

Remark 1.4. In the previous articles [24], [6] was imposed on the function $\Phi(\cdot)$ the following Δ^2 condition:

$$\Phi(x)\Phi(y) \leq \Phi(K(x+y)), \quad \exists K = \text{const} \in (1, \infty), \quad x, y \geq 0$$

or equally

$$\sup_{x, y > 0} \left[\frac{\Phi^{-1}(xy)}{\Phi^{-1}(x) + \Phi^{-1}(y)} \right] < \infty. \quad (1.5)$$

We do not suppose this condition. For instance, we can consider the function of a view $\Phi(z) = |z|^p$, which does not satisfy (1.5).

Remark 1.5. In the works of M.Ledoux and M.Talagrand [25], [40] - [42] was investigated at first the case when the function $d = d(x_1, x_2)$ is *ultrametric*, i.e. satisfies the condition

$$d(x_1, x_3) \leq \max(d(x_1, x_2), d(x_2, x_3)).$$

Note that in the classical monograph of N.Bourbaki [8], chapter 3, section 5 these function are called pseudometric.

We do not use this approach.

2 Main result: Continuity of random fields.

General Orlicz approach.

Let $\xi = \xi(x)$, $x \in X$ be separable continuous *in probability* random field (r.f), not necessary to be Gaussian. The correspondent set of elementary events, probability and expectation we will denote by ω , \mathbf{P} , \mathbf{E} , and the probabilistic Lebesgue-Riesz L_p norm of a random variable (r.v) η we will denote as follows:

$$|\eta|_p \stackrel{def}{=} [\mathbf{E}|\eta|^p]^{1/p}.$$

We find in this section some sufficient condition for continuity of $\xi(x)$ and estimates for its modulus of continuity $\omega(f, \delta)$:

$$\omega(f, \delta) = \omega(f, \delta, d) := \sup_{x_1, x_2 \in X, d(x_1, x_2) \leq \delta} w(f(x_1), f(x_2)). \quad (2.0)$$

Recall that the first publication about fractional Sobolev's inequalities [15] was devoted in particular to the such a problem; see also articles [18], [31], [38].

Let $\Phi = \Phi(u)$ be again the Young-Orlicz function. We will denote the Orlicz norm by means of the function Φ of a r.v. κ defined on our probabilistic space (Ω, \mathbf{P}) as $|||\kappa|||L(\Omega, \Phi)$ or for simplicity $|||\kappa|||\Phi$.

We introduce the so-called *natural* distance $\rho_\Phi(x_1, x_2)$ as follows:

$$d := d_\Phi = d_\Phi(x_1, x_2) := |||\rho(\xi(x_1), \xi(x_2))|||L(\Omega, \Phi), \quad x_1, x_2 \in X. \quad (2.1)$$

Theorem 2.1. *Let $m(\cdot)$ be the probabilistic minorizing measure on the set X relative the distance $d_\Phi(\cdot, \cdot)$. There exists a non-negative random variable $Z = Z(d_\Phi, m)$ with unit expectation: $\mathbf{E}Z = 1$ for which*

$$\rho(\xi(x_1), \xi(x_2)) \leq w(x_1, x_2; Z(d_\Phi, m)). \quad (2.2)$$

As a consequence: the r.f. $\xi = \xi(x)$ is d - continuous with probability one.

Proof. We pick

$$Z = \int_X \int_X \Phi \left(\frac{\rho(\xi(x_1), \xi(x_2))}{d_\Phi(x_1, x_2)} \right) m(dx_1) m(dx_2).$$

We have by means of theorem Fatou - Tonelli

$$\begin{aligned} \mathbf{E}Z &= \mathbf{E} \int_X \int_X \Phi \left(\frac{\rho(\xi(x_1), \xi(x_2))}{d_\Phi(x_1, x_2)} \right) m(dx_1) m(dx_2) = \\ &= \int_X \int_X \mathbf{E} \Phi \left(\frac{\rho(\xi(x_1), \xi(x_2))}{d_\Phi(x_1, x_2)} \right) m(dx_1) m(dx_2) = 1, \end{aligned} \quad (2.3)$$

since $\int_X \int_X m(dx_1) m(dx_2) = 1$.

It remains to apply the L.Arnold and P.Imkeller inequality.

Corollary 2.1. *Under at the same conditions as in theorem 2.1*

$$\rho(\xi(x_1), \xi(x_2)) \leq \inf_{m \in \mathcal{M}} w(x_1, x_2; Z(d_\Phi, m)). \quad (2.4)$$

where the non-negative r.v. $Z(d_\Phi, m)$ has unit expectation.

Examples.

Example 2.1. Lebesgue-Riesz spaces approach.

Suppose the measure m and distance d are such that

$$|\rho(\xi(x_1), \xi(x_2))|_p \leq d(x_1, x_2), \quad p = \text{const} \geq 1, \quad (2.5)$$

$$m^2(B(r, x)) \geq r^\theta / C(\theta), \quad r \in [0, 1], \quad \theta = \text{const} > 0, \quad C(\theta) \in (0, \infty). \quad (2.6)$$

Let also $p = \text{const} > \theta$.

Proposition 2.1. *We get using the inference of theorem 2.1 that for the r.f. $\xi = \xi(x)$ the following inequality holds: $m \in \mathcal{M}$ and*

$$\rho(\xi(x_1), \xi(x_2)) \leq 12 Z^{1/p} 4^{1/p} C^{1/p}(\theta) \frac{d^{1-\theta/p}(x_1, x_2)}{1 - \theta/p}, \quad (2.7)$$

where the r.v. Z has unit expectation: $\mathbf{E}Z = 1$.

Example 2.2. Grand Lebesgue spaces approach.

We recall first of all briefly the definition and some simple properties of the so-called Grand Lebesgue spaces; more detail investigation of these spaces see in [14], [20], [22], [26], [28], [29]; see also reference therein.

Recently appear the so-called Grand Lebesgue Spaces $GLS = G(\psi) = G\psi = G(\psi; A, B)$, $A, B = \text{const}$, $A \geq 1$, $A < B \leq \infty$, spaces consisting on all the random variables (measurable functions) $f : \Omega \rightarrow R$ with finite norms

$$\|f\|G(\psi) \stackrel{def}{=} \sup_{p \in (A, B)} [|f|_p / \psi(p)].$$

Here $\psi(\cdot)$ is some continuous positive on the *open* interval (A, B) function such that

$$\inf_{p \in (A, B)} \psi(p) > 0, \quad \psi(p) = \infty, \quad p \notin (A, B).$$

We will denote

$$\text{supp}(\psi) \stackrel{def}{=} (A, B) = \{p : \psi(p) < \infty, \}$$

The set of all ψ functions with support $\text{supp}(\psi) = (A, B)$ will be denoted by $\Psi(A, B)$.

This spaces are rearrangement invariant, see [7], and are used, for example, in the theory of probability [22], [28], [29]; theory of Partial Differential Equations [14], [20]; functional analysis [14], [20], [26], [29]; theory of Fourier series, theory of martingales, mathematical statistics, theory of approximation etc.

Notice that in the case when $\psi(\cdot) \in \Psi(A, \infty)$ and a function $p \rightarrow p \cdot \log \psi(p)$ is convex, then the space $G\psi$ coincides with some *exponential* Orlicz space.

Conversely, if $B < \infty$, then the space $G\psi(A, B)$ does not coincides with the classical rearrangement invariant spaces: Orlicz, Lorentz, Marcinkiewicz etc.

The fundamental function of these spaces $\phi(G(\psi), \delta) = \|I_A\|G(\psi), \text{mes}(A) = \delta, \delta > 0$, where I_A denotes as ordinary the indicator function of the measurable set A , by the formulae

$$\phi(G(\psi), \delta) = \sup_{p \in \text{supp}(\psi)} \left[\frac{\delta^{1/p}}{\psi(p)} \right]. \quad (2.8)$$

The fundamental function of arbitrary rearrangement invariant spaces plays very important role in functional analysis, theory of Fourier series and transform [7] as well as in our further narration.

Many examples of fundamental functions for some $G\psi$ spaces are calculated in [28], [29].

Remark 2.1 If we introduce the *discontinuous* function

$$\psi_{(r)}(p) = 1, \quad p = r; \quad \psi_{(r)}(p) = \infty, \quad p \neq r, \quad p, r \in (A, B)$$

and define formally $C/\infty = 0$, $C = \text{const} \in R^1$, then the norm in the space $G(\psi_r)$ coincides with the L_r norm:

$$\|f\|G(\psi_{(r)}) = |f|_r.$$

Thus, the Grand Lebesgue Spaces are direct generalization of the classical exponential Orlicz's spaces and Lebesgue spaces L_r .

Remark 2.2 The function $\psi(\cdot)$ may be generated as follows. Let $\xi = \xi(x)$ be some measurable function: $\xi : X \rightarrow R$ such that $\exists(A, B) : 1 \leq A < B \leq \infty, \forall p \in (A, B) \quad |\xi|_p < \infty$. Then we can choose

$$\psi(p) = \psi_\xi(p) = |\xi|_p.$$

Analogously let $\xi(t, \cdot) = \xi(t, x), t \in T$, T is arbitrary set, be some *family* $F = \{\xi(t, \cdot)\}$ of the measurable functions: $\forall t \in T \xi(t, \cdot) : X \rightarrow R$ such that

$$\exists(A, B) : 1 \leq A < B \leq \infty, \sup_{t \in T} |\xi(t, \cdot)|_p < \infty.$$

Then we can choose

$$\psi(p) = \psi_F(p) = \sup_{t \in T} |\xi(t, \cdot)|_p.$$

The function $\psi_F(p)$ may be called as a *natural function* for the family F . This method was used in the probability theory, more exactly, in the theory of random fields, see [22],[28], chapters 3,4.

For instance, the function $\Phi(\cdot)$ in (2.1) may be introduced by a natural way based on the family

$$F_{d,X} = \{\rho(\xi(x_1), \xi(x_2))\}, x_1, x_2 \in X.$$

Remark 2.3 Note that the so-called *exponential* Orlicz spaces are particular cases of Grand Lebesgue spaces [22], [28], p. 34-37. In detail, let the N - Young-Orlicz function has a view

$$N(u) = e^{\mu(u)},$$

where the function $u \rightarrow \mu(u)$ is convex even twice differentiable function such that

$$\lim_{u \rightarrow \infty} \mu'(u) = \infty.$$

Introduce a new function

$$\psi_{\{N\}}(x) = \exp \left\{ \frac{[\log N(e^x)]^*}{x} \right\},$$

where $g^*(\cdot)$ denotes the Young-Fenchel transform of the function g :

$$g^*(x) = \sup_y (xy - g(y)).$$

Conversely, the N - function may be calculated up to equivalence through corresponding function $\psi(\cdot)$ as follows:

$$N(u) = e^{\tilde{\psi}^*(\log |u|)}, |u| > 3; N(u) = Cu^2, |u| \leq 3; \tilde{\psi}(p) = p \log \psi(p).$$

The Orlicz's space $L(N)$ over our probabilistic space is equivalent up to sublinear norms equality with Grand Lebesgue space $G\psi_{\{N\}}$.

Remark 2.4. The theory of probabilistic *exponential* Grand Lebesgue spaces or equally exponential Orlicz spaces gives a very convenient apparatus for investigation of the r.v. with exponential decreasing tails of distributions. Namely, the non-zero r.v. η belongs to the Orlicz space $L(N)$, where $N = N(u)$ is function described in equality (1.8), if and only if

$$\mathbf{P}(\max(\eta, -\eta) > z) \leq \exp(-\mu(Cz)), \quad z > 1, \quad C = C(N(\cdot), \|\eta\|L(N)) \in (0, \infty).$$

(Orlicz's version).

Analogously may be written a Grand Lebesgue version of this inequality. In detail, if $0 < \|\eta\|G\psi < \infty$, then

$$\mathbf{P}(\max(\eta, -\eta) > z) \leq 2 \exp\left(-\tilde{\psi}(\log[z/\|\eta\|G\psi])\right), \quad z \geq \|\eta\|G\psi.$$

Conversely, if

$$\mathbf{P}(\max(\eta, -\eta) > z) \leq 2 \exp\left(-\tilde{\psi}(\log[z/K])\right), \quad z \geq K,$$

then $\|\eta\|G\psi \leq C(\psi) \cdot K$, $C(\psi) \in (0, \infty)$.

A very important subclass of the $G\psi$ spaces form the so-called $B(\phi)$ spaces.

Let $\phi = \phi(\lambda)$, $\lambda \in (-\lambda_0, \lambda_0)$, $\lambda_0 = \text{const} \in (0, \infty]$ be some even strong convex which takes positive values for positive arguments twice continuous differentiable function, such that

$$\phi(0) = 0, \quad \phi''(0) \in (0, \infty), \quad \lim_{\lambda \rightarrow \lambda_0} \phi(\lambda)/\lambda = \infty. \quad (2.9)$$

We denote the set of all these function as Φ ; $\Phi = \{\phi(\cdot)\}$.

We say that the *centered* random variable (r.v) $\xi = \xi(\omega)$ belongs to the space $B(\phi)$, if there exists some non-negative constant $\tau \geq 0$ such that

$$\forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \mathbf{E} \exp(\lambda \xi) \leq \exp[\phi(\lambda \tau)]. \quad (2.10).$$

The minimal value τ satisfying (2.10) is called a $B(\phi)$ norm of the variable ξ , write

$$\|\xi\|B(\phi) = \inf\{\tau, \tau > 0 : \forall \lambda \Rightarrow \mathbf{E} \exp(\lambda \xi) \leq \exp(\phi(\lambda \tau))\}.$$

This spaces are very convenient for the investigation of the r.v. having a exponential decreasing tail of distribution, for instance, for investigation of the limit theorem, the exponential bounds of distribution for sums of random variables, non-asymptotical properties, problem of continuous of random fields, study of Central Limit Theorem in the Banach space etc.

The space $B(\phi)$ with respect to the norm $\|\cdot\|B(\phi)$ and ordinary operations is a Banach space which is isomorphic to the subspace consisted on all the centered variables of Orlicz's space $(\Omega, F, \mathbf{P}), N(\cdot)$ with N - function

$$N(u) = \exp(\phi^*(u)) - 1, \quad \phi^*(u) = \sup_{\lambda} (\lambda u - \phi(\lambda)).$$

The transform $\phi \rightarrow \phi^*$ is called Young-Fenchel transform. The proof of considered assertion used the properties of saddle-point method and theorem of Fenchel-Morau:

$$\phi^{**} = \phi.$$

The next facts about the $B(\phi)$ spaces are proved in [22], [28], p. 19-40:

1. $\xi \in B(\phi) \Leftrightarrow \mathbf{E}\xi = 0$, and $\exists C = \text{const} > 0$,

$$U(\xi, x) \leq \exp(-\phi^*(Cx)), x \geq 0,$$

where $U(\xi, x)$ denotes in this article the *tail* of distribution of the r.v. ξ :

$$U(\xi, x) = \max(\mathbf{P}(\xi > x), \mathbf{P}(\xi < -x)), x \geq 0,$$

and this estimation is in general case asymptotically exact.

Here and further $C, C_j, C(i)$ will denote the non-essentially positive finite "constructive" constants.

The function $\phi(\cdot)$ may be "constructively" introduced by the formula

$$\phi(\lambda) = \phi_0(\lambda) \stackrel{\text{def}}{=} \log \sup_{t \in T} \mathbf{E} \exp(\lambda \xi(t)), \quad (2.11)$$

if obviously the family of the centered r.v. $\{\xi(t), t \in T\}$ satisfies the *uniform* Kramer's condition:

$$\exists \mu \in (0, \infty), \sup_{t \in T} U(\xi(t), x) \leq \exp(-\mu x), x \geq 0.$$

In this case, i.e. in the case the choice the function $\phi(\cdot)$ by the formula (2.11), we will call the function $\phi(\lambda) = \phi_0(\lambda)$ a *natural* function.

2. We define $\psi(p) = \psi_\phi(p) := p/\phi^{-1}(p)$, $p \geq 2$. It is proved that the spaces $B(\phi)$ and $G(\psi)$ coincides: $B(\phi) = G(\psi)$ (set equality) and both the norm $\|\cdot\|_{B(\phi)}$ and $\|\cdot\|_{G(\psi)}$ are equivalent: $\exists C_1 = C_1(\phi), C_2 = C_2(\phi) = \text{const} \in (0, \infty)$, $\forall \xi \in B(\phi)$

$$\|\xi\|_{G(\psi)} \leq C_1 \|\xi\|_{B(\phi)} \leq C_2 \|\xi\|_{G(\psi)}.$$

The Gaussian (more precisely, subgaussian) case considered in [15], [18], [38] may be obtained by choosing $\Phi(z) = \Phi_2(z) := \exp(z^2/2) - 1$ or equally $\psi(p) = \psi_2(p) = \sqrt{p}$. It may be considered easily more general example when $\Phi(z) = \Phi_Q(z) := \exp(|z|^Q/Q) - 1$, $Q = \text{const} > 0$; $\Leftrightarrow \psi(p) = \psi_Q(p) := p^{1/Q}$, $p \geq 1$.

In the last case the following implication holds:

$$\eta \in L(\Phi_Q), Q > 1 \Leftrightarrow U(\eta, x) \leq \exp(-C(\Phi, \eta) x^{Q'}),$$

where as usually $Q' = Q/(Q - 1)$.

Assume that the number θ , measure m , distance d , and the function $\psi = \psi(p)$ are such that $\theta > 0$, $C(\theta) = \text{const} \in (0, \infty)$;

$$(A, B) := \text{supp } \psi(\cdot), \tilde{A} := \max(A, \theta), B > \tilde{A}; \quad (2.12)$$

$$\|\rho(\xi(x_1), \xi(x_2))\|_{G\psi} \leq d(x_1, x_2); \quad (2.13)$$

$$m^2(B(r, x)) \geq r^\theta / C(\theta), r \in [0, 1], C(\theta) \in (0, \infty). \quad (2.14)$$

Define also a new function:

$$\psi_\theta(p) \stackrel{\text{def}}{=} (1 - \theta/p) \psi(p), \quad p \in (\tilde{A}, B). \quad (2.15)$$

Proposition 2.2. *Under formulated above conditions (2.12), (2.13), (2.14) we have: $m \in \mathcal{M}$ and*

$$|\rho(\xi(x_1), \xi(x_2))| G\psi \leq 12 d(x_1, x_2) \phi \left(G\psi_\theta, 4 C(\theta) d^{-\theta}(x_1, x_2) \right). \quad (2.16)$$

Proof. The condition (2.13) implies that

$$|\rho(\xi(x_1), \xi(x_2))|_p \leq \psi(p) \cdot d(x_1, x_2), \quad p \in (\tilde{A}, B).$$

We derive using proposition 2.1

$$\frac{|\rho(\xi(x_1), \xi(x_2))|_p}{12 \psi(p) d(x_1, x_2)} \leq \frac{(4C(\theta))^{1/p} d^{-\theta/p}(x_1, x_2)}{(1 - \theta/p)\psi(p)} = \frac{(4C(\theta))^{1/p} d^{-\theta/p}(x_1, x_2)}{\psi_\theta(p)}. \quad (2.17)$$

The assertion (2.16) of proposition 2.2 follows immediately on the basis of the definition of fundamental function for Grand Lebesgue spaces from (2.17) after taking supremum over p .

Remark 2.5. The case when $\psi(p) = \sqrt{p}$ appropriates to the Gaussian (more generally, subgaussian) random field $\xi(x)$. The case $\psi(p) = \exp(Cp)$ appears in the articles [1] and [19]. However, in both these cases the condition (1.5) is satisfied.

In the case $\psi(p) = \psi_{(r)}(p)$ we obtain the proposition 2.1. as a particular case.

Obtained in this section results specify and generalize ones in the articles [15], [18], [38].

Another approach to the problem of (ordinary) continuity of random fields based on the so-called generic chaining method and entropy technique with described applications see in [3], [10], [22], [25], [28], [40], [41] etc.

3 Weak compactness of random fields.

A. General result.

Let $\xi_n = \xi_n(x)$, $x \in X$, $n = 1, 2, \dots$ be a *sequence* of separable stochastic continuous random fields.

We suppose in this section that the metric space (X, d) is compact and that there exists a non-random point $x_0 \in X$ for which the one-dimensional r.v. $\xi_n(x_0)$ are tight.

This condition is satisfied if for instance $Y = R^1$, $\mathbf{E}\xi_n(x_0) = 0$ and $\sup_n \mathbf{Var} \xi_n(x_0) < \infty$.

Let $\bar{\Phi} = \bar{\Phi}(u)$ be again the Young-Orlicz function, a single function for all the r.f. $\xi_n(x)$. Define the common distance

$$\bar{d}(x_1, x_2) = \sup_n d_n(x_1, x_2), \quad x_1, x_2 \in X,$$

where

$$d_n(x_1, x_2) = \|\rho(\xi_n(x_1), \xi_n(x_2))\| L(\bar{\Phi}). \quad (3.1)$$

Suppose that the set X is compact set relative the distance $\bar{d} = \bar{d}(x_1, x_2)$.

We intend in this section to obtain the sufficient condition for weak compactness of the distributions of the family $\xi_n = \xi_n(\cdot)$ in the space of continuous functions $C(X) = C(X; \bar{d})$.

Theorem 3.1. *Let in addition $m(\cdot)$ be probabilistic minorizing measure on the set X relative the distance $\bar{d}(\cdot, \cdot)$ and the Young - Orlicz function $\bar{\Phi}$. Then the family of distributions generated by the continuous versions of r.f. $\{\xi_n(\cdot)\}$ in the space of \bar{d} continuous functions $C(X) = C(X, \bar{d})$:*

$$\nu_n(A) = \mathbf{P}(\xi_n(\cdot) \in A),$$

where A is arbitrary Borelian set in $C(X)$, is weakly compact, i.e. tight.

Proof. It follows from theorem 2.1 that there exists a sequence of non-negative random variable $Y_n = Y_n(d_{\bar{\Phi}}, m)$ with unit expectation: $\mathbf{E}Y_n = 1$ for which

$$\bar{\rho}(\xi_n(x_1), \xi_n(x_2)) \leq w(x_1, x_2; Y_n, \bar{d}, m). \quad (3.2)$$

Therefore,

$$\forall \epsilon > 0 \Rightarrow \lim_{\delta \rightarrow 0+} \mathbf{P}(\omega(\xi_n, \delta) > \epsilon) = 0. \quad (3.3)$$

This completes the proof of theorem 3.1.

Consequence 3.1. Suppose in addition to the conditions of theorem 3.1 that as $n \rightarrow \infty$ the finite-dimensional distributions of the r.f. $\xi_n(x)$ converge to the finite-dimensional distributions of some r.f. $\xi_\infty(x)$. Then the sequence of distributions $\nu_n(\cdot)$ weakly converges to the $\nu_\infty(\cdot)$. Namely, for every continuous bounded functional $F : C(X, \bar{d}) \rightarrow R$

$$\lim_{n \rightarrow \infty} \mathbf{E}F(\xi_n(\cdot)) = \mathbf{E}F(\xi_\infty(\cdot)). \quad (3.4)$$

Remark 3.1. Let $\Phi_n(u)$ be "individual" Young - Orlicz function for each field $\xi_n(\cdot)$ described below. The "common" Young - Orlicz function $\bar{\Phi}(u)$ may be constructed evidently as follows:

$$\bar{\Phi}(u) = \sup_n \Phi_n(u),$$

if it is finite for all the values u , $u \in R$.

For instance, let $\psi_n = \psi_n(p)$ be natural function for the r.f. $\xi_n(x)$; $1 \leq p \leq b_n$. Assume $b := \inf_n b_n > 1$ and suppose

$$\psi_\infty(p) := \sup_n \psi_n(p) < \infty, \quad 1 \leq p < b. \quad (3.5)$$

Then the $G\psi_\infty$ space is suitable for us; the correspondent Young - Orlicz function $\Phi_\infty = \overline{\Phi}$ is described in the second section.

Analogously may be considered the case of $B(\phi)$ spaces. The common function $\overline{\phi}(\cdot)$ may be "constructively" introduced by the formula

$$\overline{\phi}(\lambda) = \phi_\infty(\lambda) \stackrel{def}{=} \log \sup_n \sup_{x \in X} \mathbf{E} \exp(\lambda \xi_n(x)), \quad (3.6)$$

if obviously the family of the centered r.f. $\{\xi_n(x), n = 1, 2, \dots; x \in X\}$ satisfies the *uniform* Kramer's condition:

$$\exists \mu \in (0, \infty), \sup_n \sup_{x \in X} U(\xi_n(x), u) \leq \exp(-\mu u), \quad u \geq 0. \quad (3.7)$$

B. Central Limit Theorem in the space of continuous functions.

In this subsection $Y = R^1$ and ρ is any continuous distance in Y .

Let $\eta_i = \eta_i(x)$, $x \in X$ be independent centered: $\mathbf{E}\eta_i(x) = 0$ identical distributed random fields with finite covariation function $R(x_1, x_2) = \text{cov}(\eta_i(x_1), \eta_i(x_2)) = \mathbf{E}\eta_i(x_1) \cdot \eta_i(x_2)$. Denote

$$S_n(x) = n^{-1/2} \sum_{i=1}^n \eta_i(x). \quad (3.8)$$

Obviously, the finite-dimensional distributions of r.f. $S_n(\cdot)$ converge as $n \rightarrow \infty$ to the finite-dimensional distributions of the centered Gaussian r.f. $S_\infty(\cdot)$ with at the same covariation function $R(\cdot, \cdot)$.

Definition 3.1. (See [9], [22]).

We will say that the sequence of r.f. $\{\eta_i(\cdot)\}$ satisfies the Central Limit Theorem (CLT) in the space $C(X, d)$, if $\mathbf{P}(\eta_i(\cdot) \in C(X, d)) = 1$ and the distributions of the r.f. $S_n(\cdot)$ in the set $C(X, d)$ converge weakly as $n \rightarrow \infty$ to the distribution of the Gaussian r.f. $S_\infty(\cdot)$.

We formulate here some sufficient conditions for the CLT in the space of continuous functions in the terms of minorizing measures.

Note that in the terms of majorizing measures these conditions are obtained, e.g., in [17], [9], [25], [41]; in the entropy terms - in [22], [28], chapter 4, section 4 etc.

We can use the the result of the last subsection. Namely, let $\zeta = \zeta(\lambda)$ be natural ϕ - function for the r.f. $\eta_1(x)$:

$$\zeta(\lambda) := \sup_{x \in X} \log \mathbf{E} \exp(\lambda \eta_1(x)), \quad (3.9)$$

if as before the family of the centered r.v. $\{\eta_1(x), x \in X\}$ satisfies the uniform Kramer's condition.

We have:

$$\log \mathbf{E} \exp(\lambda S_n(x)) \leq n^{-1/2} \zeta(\lambda/n) \leq \bar{\zeta}(\lambda), \quad (3.10)$$

where by definition

$$\bar{\zeta}(\lambda) \stackrel{def}{=} \sup_n [n^{-1/2} \zeta(\lambda/n)] < \infty, \quad |\lambda| < \lambda_0 = \text{const} > 0. \quad (3.11)$$

For instance, let

$$\zeta(\lambda) = \lambda^2 I(|\lambda| \leq 1) + |\lambda|^Q I(|\lambda| > 1), \quad Q = \text{const} \geq 1,$$

then

$$\bar{\zeta}(\lambda) \asymp \lambda^2 I(|\lambda| \leq 1) + |\lambda|^{\max(Q,2)} I(|\lambda| > 1).$$

The equivalent conclusion in the terms of Φ - functions: if here

$$\Phi(u) = \exp(|u|^\beta) - 1, \quad |u| \geq 1, \quad \beta = \text{const} > 1,$$

then

$$\bar{\Phi}(u) \asymp \exp(C |u|^{\max(\beta', 2)'}) - 1, \quad |u| \geq 1.$$

Introduce also the Young - Orlicz function

$$\Theta(u) = e^{\bar{\zeta}^*(u)} - 1$$

and the correspondent distance

$$\theta(x_1, x_2) = \|\eta_1(x_1) - \eta_1(x_2)\| L(\Theta).$$

Theorem 3.2. *Let $m(\cdot)$ be any probabilistic minorizing measure on the set X relative the distance $\theta(\cdot, \cdot)$ and the Young - Orlicz function Θ . Then the sequence of r.f. $\{\eta_i(\cdot)\}$ satisfies the Central Limit Theorem (CLT) in the space $C(X, \theta)$.*

Proof follows immediately from the theorem 3.1., where we set $\xi_n(x) = S_n(x)$. It remains to ground only the weak compactness of the family r.f. $S_n(\cdot)$. Note that

$$\mathbf{E} e^{\lambda S_n(x)} \leq e^{\bar{\zeta}(\lambda)},$$

or equally

$$\sup_n \sup_{x \in X} \|S_n(x)\| L(\Theta) < \infty, \quad (3.12)$$

see [28], chapter 1, section 2. Analogously,

$$\sup_n \|S_n(x_1) - S_n(x_2)\| L(\Theta) \leq C \cdot \theta(x_1, x_2). \quad (3.13)$$

This completes the proof of theorem 3.2; see, e.g. [37].

Another approach used the conception of $G\psi$ spaces. In detail, introduce the natural ψ function as described below:

$$\psi(p) = \sup_{x \in X} |\eta_1(x)|_p,$$

and suppose $\exists p > 2$ such that $\psi(p) < \infty$.

The following distance is finite:

$$d_\psi(x_1, x_2) = \|\eta_1(x_1) - \eta_1(x_2)\|_{G\psi}.$$

Define a new ψ function

$$\bar{\psi}(p) = \left\lceil \frac{p}{\log(p+1)} \right\rceil \cdot \psi(p).$$

It follows from the famous Rosenthal inequality that

$$\sup_{x \in X} \sup_n \|S_n\|_{G\bar{\psi}} < \infty$$

and

$$\sup_n \|S_n(x_1) - S_n(x_2)\|_{G\bar{\psi}} \leq C \cdot d_\psi(x_1, x_2).$$

It remains to use the proposition of theorem 3.1.

4 Non-asymptotical estimates of maximum for random fields.

Grand Lebesgue spaces approach.

Let $\xi = \xi(x)$, $x \in X$ be again separable random field (or process) with values in the real axis R , $T = \{x\}$ be arbitrary Borelian subset of X .

Denote

$$\bar{\xi}_T = \sup_{x \in T} \xi(x), \quad \bar{\xi} = \bar{\xi}_X = \sup_{x \in X} \xi(x).$$

Proposition 4.1. Let all the notation of proposition 2.2 be retained and condition be satisfied. Denote also

$$D = \text{diam}(X, d) = \sup_{x_1, x_2 \in X} d(x_1, x_2).$$

We assert:

$$\|\bar{\xi}\|_{G\psi} \leq \inf_{x_0 \in X} \|\xi(x_0)\|_{G\psi} + 12 D \cdot \phi\left(G\psi_\theta, 4C(\theta)D^{-\theta}\right). \quad (4.1)$$

Proof. Let x_0 be arbitrary point in the set X ; we have

$$\xi(x) = \xi(x_0) + (\xi(x) - \xi(x_0)) \leq \xi(x_0) + \rho(\xi(x_0), \xi(x)),$$

$$\sup_x \xi(x) \leq \xi(x_0) + \sup_x \rho(\xi(x_0), \xi(x)) \leq \xi(x_0) + \sup_{x_1, x_2} \rho(\xi(x_1), \xi(x_2)).$$

We conclude using triangle inequality for $G\psi$ norm and inequality 2.16:

$$||\bar{\xi}||G\psi \leq ||\xi(x_0)||G\psi + 12D \cdot \phi(G\psi_\theta, 4C(\theta)D^{-\theta}).$$

Since the value x_0 is arbitrary, we convince itself that estimate (4.1) is true.

Example 4.1. Let us consider as a particular case the possibility $\psi(p) = \psi_{(r)}(p)$, where $r = \text{const} > \max(\theta, 1)$. We get:

$$|\bar{\xi}|_r \leq \inf_{x_0 \in X} |\xi(x_0)|_r + 12 \cdot \frac{(4C(\theta))^{1/r} \cdot D^{1-\theta/r}}{1 - \theta/r}. \quad (4.2)$$

Here

$$d(x_1, x_2) = d_r(x_1, x_2) = |\xi(x_1) - \xi(x_2)|_r. \quad (4.3)$$

Exponential estimations.

Let $\Phi = \Phi(u)$ be again as in the beginning of the second section the Young-Orlicz function generated by the r.f. $\xi(x)$, so that

$$\sup_{x \in X} |||\xi(x)|||L(\Phi) < \infty;$$

in the sequel we will conclude without loss of generality

$$\sup_{x \in X} |||\xi(x)|||L(\Phi) = 1. \quad (4.4)$$

Recall that we will denote the Orlicz norm by means of the function Φ of a r.v. κ defined on our probabilistic space as $|||\kappa|||L(\Phi)$.

We introduce the natural distance $\rho_\Phi(x_1, x_2)$ as follows:

$$d_\Phi(x_1, x_2) := |||\rho(\xi(x_1), \xi(x_2))|||L(\Phi), \quad x_1, x_2 \in X; \quad (4.5)$$

then in particular

$$D = D(\Phi) := \text{diam}(X, d_\Phi) \leq 2.$$

For arbitrary Borelian subset $T \subset X$ we denote

$$Q(T, u) = \mathbf{P}(\sup_{t \in T} \xi(t) > u), \quad u \geq 2. \quad (4.6)$$

$$Q_+(T, u) = \mathbf{P}(\sup_{t \in S} |\xi(t)| > u), \quad u \geq 2. \quad (4.7)$$

Our purpose in the rest of this section is obtaining an exponentially exact as $u \rightarrow \infty$, $u > u_0 = \text{const} > 0$ estimation for the probability $Q(u) \stackrel{\text{def}}{=} Q(X, u)$, $Q_+(u) \stackrel{\text{def}}{=} Q_+(X, u)$ in the terms of "minorizing measures" and $B(\phi)$ spaces.

In the entropy terms this problem is considered in [9], [11], [12], [28], chapter 3, [33]: in the terms of majorizing measures- in [25], [32], [40] etc.

The estimations of $Q_+(u) \stackrel{\text{def}}{=} Q_+(X, u)$ are used in the Monte-Carlo method, statistics, numerical methods etc., see [13], [16], [28], [35], [36].

For instance, we can suppose that the random field $\xi(x)$ to be centered and satisfies the uniform Kramer's condition, so that the natural function

$$\phi(\lambda) = \log \sup_{x \in X} \mathbf{E} \exp(\lambda \xi(x))$$

is finite in some non-trivial interval $\lambda \in (-\lambda_0, \lambda_0)$, $\lambda_0 = \text{const} \in (0, \infty]$.

Then we may introduce the following Young-Orlicz function (up to multiplicative positive constant)

$$\Phi_\phi(u) = \exp(\phi^*(u)) - 1,$$

so that $\sup_{x \in X} \|\xi(x)\| B(\phi) = 1$ and following $\sup_x \|\xi(x)\|(\Phi) < \infty$.

We need also to suppose that the function $\Phi = \Phi(u)$ satisfies the condition (1.5) with finite non-zero constant K .

Let us introduce the following constant (more exactly, functional)

$$C_2 = C_2(\Phi) = \frac{\Phi^{-1}(1)}{54K^2}, \quad (4.8)$$

and define by $N(T) = N(T, w, \epsilon)$, $\epsilon > 0$ as usually for the (pre-compact) metric set (T, w) , $T \subset X$ the minimal number of closed balls with radii ϵ : $B(x_j, \epsilon) = B(x_j, w, \epsilon) = \{x_1 : w(x, x_1) \leq \epsilon\}$ which cover the set T :

$$T \subset \cup_{j=1}^{N(T)} B(x_j, w, \epsilon); \quad N(\epsilon) := N(X, w, \epsilon). \quad (4.9)$$

Recall that the logarithm of $N(X, w, \epsilon)$

$$H(X, w, \epsilon) = \log N(X, w, \epsilon)$$

is called "entropy" of the set X relative the distance $w(\cdot, \cdot)$ and widely used in the entropy approach to the investigation of continuity of random processes and fields.

Proposition 4.2. *Under formulated above conditions*

$$Q(u) \leq \inf_{\delta \in (0, D)} \frac{N(X, w, \delta)}{\Phi(u/(1 + \delta/C_2(\Phi)))}, \quad (4.10)$$

$$Q_+(u) \leq 2 \inf_{\delta \in (0, D)} \frac{N(X, w, \delta)}{\Phi(u/(1 + \delta/C_2(\Phi)))}, \quad (4.11)$$

Proof.

1. S.Kwapien and J.Rosinsky proved in [24] the following inequality:

$$\mathbf{E}\Phi \left(2 C_2 \sup_{t \neq s} \frac{(\xi(t) - \xi(s))}{w(t, s)} \right) \leq 1 + \sup_{t \neq s} \mathbf{E}\Phi \left(\frac{(\xi(t) - \xi(s))}{d(t, s)} \right). \quad (4.12)$$

As long as we choose $d(t, s) = d_\Phi(t, s)$, we have

$$\mathbf{E}\Phi \left(2 C_2 \sup_{t \neq s} \frac{(\xi(t) - \xi(s))}{w(t, s)} \right) \leq 2. \quad (4.13)$$

Recall that $\Phi = \Phi(u)$ is convex function and $\Phi(0) = 0$; following

$$\Phi \left(\frac{u}{2} \right) = \Phi \left(\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot u \right) \leq \frac{1}{2} \Phi(0) + \frac{1}{2} \Phi(u) = \frac{1}{2} \Phi(u),$$

We conclude on the basis of inequality (4.13)

$$\mathbf{E}\Phi \left(C_2 \sup_{t \neq s} \frac{(\xi(t) - \xi(s))}{w(t, s)} \right) \leq 1, \quad (4.14)$$

or equally

$$||| \sup_{w(x_1, x_2) \leq \delta} (\xi(x_1) - \xi(x_2)) ||| \Phi \leq \delta / C_2. \quad (4.15)$$

2. Let x_0 be fixed (non-random) point in the set X . Consider the ball $B = B(x_0, w, \delta) = \{x_1, x_1 \in X, w(x_0, x_1) \leq \delta\}$, $0 < \delta \leq D$. We have for the values $x \in B$ using triangle inequality and (4.15): $\xi(x) = \xi(x_0) + [\xi(x) - \xi(x_0)]$;

$$\begin{aligned} \sup_{x \in B} \xi(x) &\leq \xi(x_0) + \sup_{w(x_1, x_2) \leq \delta} (\xi(x_1) - \xi(x_2)), \\ ||| \sup_{x \in B} \xi(x) ||| \Phi &\leq ||| \xi(x_0) ||| \Phi + ||| \sup_{w(x_1, x_2) \leq \delta} (\xi(x_1) - \xi(x_2)) ||| \Phi \leq \\ &1 + \delta / C_2. \end{aligned} \quad (4.16)$$

It follows from Tchebychev's inequality

$$Q(B(x_0, w, \delta), u) \leq 1 / \Phi(u / (1 + \delta / C_2)). \quad (4.17)$$

3. The first assertion of proposition 4.2 follows now immediately:

$$\begin{aligned} Q(u) &= \mathbf{P} \left[\bigcup_{j=1}^{N(T, w, \delta)} \left\{ \sup_{x \in B(x_j, w, \delta)} \xi(x) > u \right\} \right] \leq \\ &\sum_{j=1}^{N(T, w, \delta)} \mathbf{P} \left[\left\{ \sup_{x \in B(x_j, w, \delta)} \xi(x) > u \right\} \right] \leq N(T, w, \delta) \cdot [1 / \Phi(u / (1 + \delta / C_2))] \end{aligned} \quad (4.18)$$

after minimization over δ .

The second assertion of proposition 4.2 follows from the inequality

$$Q_+(u) \leq \mathbf{P}(\sup_{x \in X} \xi(x) > u) + \mathbf{P}(\sup_{x \in X} (-\xi(x)) > u).$$

Examples.

Example 4.1. Suppose in addition to the conditions of proposition 4.2 that the function $u \rightarrow \Phi(u)$, $u > 0$ is logarithmical convex:

$$(\log \Phi)''(u) > 0.$$

Let also $\gamma = \text{const} \in (0, 1)$. Denote

$$\delta_0 = \delta_0(u; \gamma, \Phi) = \frac{C_2 \gamma}{u \cdot [\log \Phi]'(u)}. \quad (4.19)$$

We obtain choosing $\delta = \delta_0$ substituting into (4.10) that for all sufficiently greatest values u : $\delta_0(u; \gamma, \Phi) < D$

$$Q(u) \leq \frac{(1 - \gamma)^{-1}}{\Phi(u)} \cdot N \left(\frac{C_2 \gamma}{u \cdot [\log \Phi]'(u)} \right). \quad (4.20)$$

Example 4.2. Let now $\Phi(u) = \exp(u^2/2) - 1$ (subgaussian case). Suppose

$$N(\epsilon) \leq C_3 \epsilon^{-\kappa}, \quad \epsilon \in (0, D), \quad \kappa = \text{const} > 0. \quad (4.21)$$

The value κ is said to be *majorital* dimension of the set X relative the distance w .

The optimal value γ in (4.20) is equal to $\gamma = \gamma_0 := \kappa/(\kappa + 1)$ and we conclude for the values U such that

$$\delta_0 = \frac{C_2 \kappa}{(\kappa + 1)u^2} \leq D :$$

$$Q(u) \leq C_3 C_2^{-\kappa} \kappa^{-\kappa} (\kappa + 1)^{\kappa+1} u^{2\kappa} e^{-u^2/2}. \quad (4.22)$$

Example 4.3. Assume that in the example 4.2 instead the condition (4.21) the following condition holds:

$$N(\epsilon) \asymp C_4 e^{C_5 \epsilon^{-\beta}}, \quad \epsilon \in (0, D); \quad \beta = \text{const} > 0. \quad (4.23)$$

Then

$$Q(u) \leq e^{-0.5u^2 + C_6 u^{2\beta/(\beta+1)}}, \quad u \geq C_7. \quad (4.24)$$

Note that in the case $\beta \geq 2$ the so-called entropy series

$$\sum_{n=1}^{\infty} 2^{-n} H^{1/2}(X, w, 2^{-n})$$

diverges.

5 Concluding remarks.

A. Degrees.

Let $X = [0, 1]^n$ $n = 2, 3, \dots$. In the articles [38], [18] is obtained a multivariate generalization of famous Garsia-Rodemich-Rumsey inequality [15]. Roughly speaking, instead degree "2" in the inequalities (1.3) and (1.4) stands degree 1 and coefficients dependent on d .

The ultimate value of this degree in general case of arbitrary metric space (X, d) is now unknown; see also [1], [19].

B. Spaces.

Notice that in all considered cases and under our conditions when $\sup_x \|\xi(x)\| < \infty$, then $\|\sup_x \xi(x)\| < \infty$. But in the article [34] was constructed a "counterexample": there exists a continuous a.e random process for which

$$\sup_x \|\xi(x)\| < \infty, \quad \|\sup_x \xi(x)\| = \infty.$$

This circumstance imply that our conditions are only sufficient but not necessary.

C. Rectangle distance.

In the report [31] for the multivariate functions was introduced the rectangle distance. For instance, if the r.f. $\xi = \xi(x, y)$, $x, y \in [0, 1]$ is bivariate, then

$$r_\xi(x_1, x_2; y_1, y_2) = \|\xi(x_2, y_2) - \xi(x_1, y_2) - \xi(x_1, y_1) + \xi(x_1, y_1)\|.$$

It is very interest by our opinion to obtain in general case estimations for $r_\xi(x_1, x_2; y_1, y_2)$ in the terms of minorizing measures alike the one-variate case considered here.

D. Lower bounds.

The lower estimates for probabilities $Q(u)$ are obtained e.g. in [28], chapter 3, sections 3.5-3.8. They are obtained in entropy terms, all the more so in the terms of minorizing measures.

Note that the lower bounds in [28] may coincide up to multiplicative constants with upper bounds.

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